

Kerdock Codes for Limited Feedback Precoded MIMO Systems

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Abstract

A codebook based limited feedback strategy is a practical way to obtain partial channel state information at the transmitter in a precoded multiple-input multiple-output (MIMO) wireless system. Conventional codebook designs use Grassmannian packing, equiangular frames, vector quantization, or Fourier based constructions. While the capacity and error rate performance of conventional codebook constructions have been extensively investigated, constructing these codebooks is notoriously difficult relying on techniques such as nonlinear search or iterative algorithms. Further, the resulting codebooks may not have a systematic structure to facilitate storage of the codebook and low search complexity. In this paper, we propose a new systematic codebook design based on Kerdock codes and mutually unbiased bases. The proposed Kerdock codebook consists of multiple mutually unbiased unitary bases matrices with quaternary entries and the identity matrix. We propose to derive the beamforming and precoding codebooks from this base codebook, eliminating the requirement to store multiple codebooks. The proposed structure requires little memory to store and, as we show, the quaternary structure facilitates codeword search. We derive the chordal distance for two antenna and four antenna codebooks, showing that the proposed codebooks compare favorably with prior designs. Monte Carlo simulations are used to compare achievable rates and error rates for different codebooks sizes.

Index Terms

Array signal processing, MIMO systems, feedback communication, codes.

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I. INTRODUCTION

Multiple-input multiple-output (MIMO) systems can provide considerable capacity and resilience to channel fading, especially if channel state information is available at the transmitter (CSIT) [1]. A practical solution to provide channel state information (CSI) to the transmitter is a codebook based feedback strategy, known as limited feedback or finite rate feedback [2]–[6]. Using limited feedback, the receiver computes the appropriate transmit precoder from a finite set of precoders, called the *codebook*, shared by the transmitter and the receiver. The receiver then sends the index of the codeword (often in the order of few bits) back to the transmitter resulting in the feedback of quantized CSI. The amount of feedback depends on the size of the codebook and the feedback rate. The size of the codebook severely affects the system performance [2], [7]. Smaller codebooks reduce the feedback bandwidth but result in coarse quantization and reduced system performance. Larger codebooks approach the ideal CSIT case, though at an exponentially decaying rate (*i.e.* diminishing returns) [7]. In practice, 3 to 6 bits of feedback are common to balance the performance and feedback tradeoff [8]. A significant practical issue with limited feedback is the storage and search complexity associated with the codebook [9]. Without special structure, multiple codebooks must be stored element by element with different codebooks for beamforming and spatial multiplexing with different numbers of streams. Codeword search at the receiver grows linearly with codebook size. In this paper, we propose a systematically constructed finite alphabet codebook that yields favorable system performance with additional benefit of reduced storage and search complexity.

A. Background

One of the first approaches for codebook design suggested by Narula *et al.* [2] is inspired by the Lloyd algorithm from the vector quantization (VQ) literature [10]. VQ based codebook design methods have subsequently been extensively investigated in [11]–[14]. Roh and Rao [11] proposed a mean-squared weighted inner product (MSwIP) criteria to obtain a closed form centroid solution and find optimal codebooks for multiple-input single-output (MISO) system. They extended the results to MIMO systems in [12] to obtain codebooks that minimize capacity loss. A variation of the VQ method by constraining the quantization space to unit hypersphere (*i.e.* sphere vector quantization) was proposed by Xia and Giannakis [13]. The VQ based approach with the Lloyd algorithm has been shown to provide asymptotic improvement in capacity loss

with increasing codebook size [3], [5], [15]–[17]. The Lloyd algorithm requires a large number of iterations that increase with the number of transmit antennas. Thus, VQ is primarily suitable for offline design and analytical comparisons. Furthermore, the VQ based codebook does not yield any structure in the codebook. Each entry, often complex valued, with maximum available precision must be stored. For codebook selection at the receiver, an exhaustive search with complex matrix operation is usually required [9].

The Grassmannian packing approach for codebook design was proposed for beamforming by Love *et al.* [4] and Muekkavilli *et al.* [3], then later extended to precoding by Love and Heath [6]. The problem of Grassmannian packing is to find a set of subspaces, which are points in the Grassmann manifold, and are maximally spaced in terms of subspace distance. Quantization on the Grassmann manifold has been studied by Mondal *et al.* [7], [18]. A major challenge associated with Grassmannian codebooks is that it is difficult to find good packings and thus good codebooks. The real case has been studied from an algebraic point of view by Conway *et al.* [19] and Shor and Sloane [20], while bounds on maximum minimum distance were investigated by Bachoc [21] and Barg and Nogin [22]. An extensive computer search is usually required (i.e. offline design) and the resulting codebook generally does not have any structure to ease the storage and search computation requirement [23].

Mondal *et al.* proposed an equiangular frame (EF) based codebook using mutual information criteria and showed that (at least in the real case) the EF codebook maximizes the packing radius if and only if it is an EF, and that EF codebooks achieve lower average distortion compared to VQ or Grassmannian codebooks [24]. The EF codebook is based on Grassmannian frames proposed by Strohmer and Heath [25], and are also connected with Grassmannian packings, spherical designs, and regular graphs. As with Grassmannian packing, constructing equiangular frames is also challenging. Tropp *et al.* proposed an alternating projection method to construct equiangular frames [26]. Alternating projection algorithm iteratively projects onto two constraint sets, a unit norm constraint set and unit tight frame constraint set, to arrive at the approximate equiangular frame with prescribed error. Unfortunately, apart from the equiangular distance property, the resulting codebook does not have any constructive structure and the alternating projection algorithm does not work well for small finite alphabets [26].

One popular codebook is the Fourier codebook proposed in [4], [6], inspired by the unitary space-time constellation design in Hochwald *et al.* [27]. Exhaustive search is used to find a set of

Fourier matrices, and modifications of Fourier matrices by a diagonal generator matrix of complex exponentials that maximize the minimum correlation between codewords. This construction is being considered for 3GPP long-term evolution (LTE) and 3GPP2 ultra mobile broadband (UMB) due to its systematic codebook generation [8]. A connection between the Fourier based design, the Welch bound [28], and difference sets were reported in [29]. For practical applications, the Fourier based design is attractive because only two matrices, the generator matrix and the discrete Fourier transform (DFT) matrix, need to be stored. Codeword search is somewhat simplified at the receiver since it can exploit the structure of the DFT matrix, but matrix computations with complex arithmetic are still required because the codebook contains complex values.

To minimize storage requirements and search complexity, a quadrature amplitude modulation (QAM) based codebook was proposed by Ryan *et al.* [9]. The codebook design was motivated by limiting the codeword entries to a finite alphabet (i.e. QAM constellation points) and reducing the search complexity using maximum likelihood non-coherent lattice decoding [30]. This method is essentially an element by element quantization of the precoder matrix to the nearest QAM points. The main drawback is that the resulting feedback rate is larger than other codebook constructions, which makes it less attractive in practice.

B. Contributions

In this paper, we propose a single user codebook design for limited feedback MIMO systems based on Kerdock codes. Kerdock codes were originally proposed for error correction [31]. It was shown by Hammons *et al.* [32] that Kerdock codes are \mathbb{Z}_4 (integer modulo 4 or quaternary alphabet) linear containing more codewords than any known linear code with the same minimal distance. We exploit the connection between Kerdock codes and mutually unbiased bases (see e.g. [33] for details). Our proposed Kerdock codebook consists of multiple mutually unbiased unitary bases matrices with quaternary entries and the identity matrix. This codebook is used to derive codebooks for beamforming and precoding. The beamforming codebook is derived from all the columns of the codebook while the precoding codebooks are derived by subsets of columns from each bases. Note that the inclusion of the identity matrix, which is not ad hoc but actually forms one of the mutually unbiased bases, means that antenna subset selection [34] is included as a special case. We consider two practical examples of codebooks for two and four transmit antennas using two different constructions: a Sylvester-Hadamard construction [33] and

a power construction [35]. The Sylvester-Hadamard construction gives a good solution for the two antenna case while the structure in the power construction gives a better solution for the four antenna case. The power construction also facilitates closed form derivation of certain subspace distance properties. While we are mainly concerned with narrowband single user MIMO system in this paper, multiuser MIMO [36] and wideband systems [37] appears to be promising.

The application of Kerdock codes is motivated by the subspace distance property with the additional practical benefit of storage and search complexity reduction. We demonstrate the benefits of the proposed Kerdock codebook using the following metrics:

- 1) system performance,
- 2) proximity to a Grassmannian codebook,
- 3) construction & storage,
- 4) search efficiency.

To evaluate system performance, we compare the vector symbol error rate (VSER) and the achievable rate with different codebooks and with perfect CSIT. Our Monte Carlo simulation results show that Kerdock codebooks have negligible performance loss, and in some cases performance gain, with previously proposed codebooks. The proposed Kerdock codebook satisfies the sufficient conditions in [38], and thus is full-diversity. To measure proximity to a Grassmannian codebook, we compute the maximum minimum subspace distance (in this case we use the Fubini-Study distance) and compare with the best known Grassmannian packings. Because of the special structure of our codebook, we are able to derive exactly the Fubini-study distance for our two and four antenna codebooks for all dimensions of precoders in closed form. This special structure shows that the proposed Kerdock codebooks are quite close to Grassmannian codebooks. To evaluate construction and storage, we estimate the required storage as a function of the number of bits of precision including storing multiple codebooks for different modes of transmission (*i.e.* beamforming and spatial multiplexing). The storage required for Kerdock codes is much smaller than for general Grassmannian, VQ, or EF codebooks and is somewhat smaller than the Fourier construction due to the quaternary structure. Finally, we estimate the search efficiency by estimating the number of operations required to find the optimum codeword in the codebook at the receiver. Here the main simplification comes from the fact that the entries of the codebook are either scaled version of $\{1, -1, j, -j\}$ or are zero. Consequently the multiplication

operations become simple sign flipping or flipping real and imaginary components.

C. Organization

This paper is organized as follows. In Section II, an overview of the system model is given. In Section III, the construction of the MUB codebook is outlined and the strategies for precoded MIMO systems are given. In Section IV, we analyze the storage and search complexity compared with Grassmannian and Fourier based design. In Section V, we analyze the distance, diversity, and capacity performance. In Section VI, we provide numerical simulation results to support our analysis. Finally, we conclude this paper with some remarks in Section VII.

For notation, we use lower case bold letters, *e.g.* \mathbf{v} , to denote vectors and upper case bold letters, *e.g.* \mathbf{H} , to denote matrices. The $n \times n$ identity matrix is denoted by \mathbf{I}_n . The space of real and complex are denoted by \mathbb{R} and \mathbb{C} , respectively with an appropriate superscript to denote the dimension of the respective spaces. We shall use T and * to denote the transposition and Hermitian transpose, respectively.

II. SYSTEM OVERVIEW

A. Discrete-time System Model

A limited feedback precoded MIMO wireless system with M_t transmit antennas and M_r receive antennas is shown in Fig. 1. The transmit bit stream is sent to the encoder and modulator, which outputs a complex transmit vector, $\mathbf{s}[k] = [s_1[k], s_2[k], \dots, s_{M_s}[k]]^T$, where k denotes the time index and M_s denotes the number of streams to be sent. Note that beamforming is the special case where $M_s = 1$, and $1 < M_s \leq M_t$ for M_s -stream spatial multiplexing. We assume that $E_s[\mathbf{s}\mathbf{s}^*] = \frac{\mathcal{E}_s}{M_s} \mathbf{I}_{M_s}$ to constrain the average transmit power, E_s is used to denote the expectation with respect to the transmit vector, and \mathcal{E}_s is used to denote the total transmit power.

The transmit vector $\mathbf{s}[k]$ is then multiplied by the unitary precoder \mathbf{F} (\mathbf{f} for beamforming) of size $M_t \times M_s$ with $\mathbf{F}^* \mathbf{F} = (1/M_s) \mathbf{I}_{M_s}$, producing a length M_t transmit vector $\mathbf{x}[k] = \sqrt{\mathcal{E}_s/M_s} \mathbf{F} \mathbf{s}[k]$. The precoder \mathbf{F} is selected based on limited feedback information from the receiver.

Assuming perfect synchronization, sampling, and a linear memoryless channel, the equivalent baseband input-output relationship can be written as

$$\mathbf{y}[k] = \sqrt{\frac{\mathcal{E}_s}{M_s}} \mathbf{H} \mathbf{F} \mathbf{s}[k] + \mathbf{n}[k] \quad (1)$$

where \mathbf{H} is the channel matrix and $\mathbf{n}[k]$ is the noise vector. We assume that \mathbf{H} is an $M_r \times M_t$ matrix where each entry is unit variance complex Gaussian independent and identically distributed (i.i.d.) according to $\mathcal{CN}(0, 1)$. The entries of $\mathbf{n}[k]$ are also complex Gaussian i.i.d. distribution according to $\mathcal{CN}(0, N_0)$. The receive vector $\mathbf{y}[k]$ is then decoded by assuming a perfect knowledge of $\mathbf{H}\mathbf{F}$ at the receiver to produce the output vector $\hat{\mathbf{s}}$.

We assume that the receiver has a perfect estimate of the channel matrix \mathbf{H} and uses a linear receiver which applies an $M_s \times M_r$ matrix \mathbf{G} to the receive symbol $\mathbf{y}[k]$. For the spatial multiplexing case, the zero-forcing (ZF) receiver is used which is given by $\mathbf{G} = (\mathbf{H}\mathbf{F})^\dagger$ where $(\cdot)^\dagger$ denotes the Moore-Penrose pseudo inverse. For the beamforming case, a maximum ratio combining (MRC) receiver is assumed with $\mathbf{G} = (\mathbf{H}\mathbf{F})^*$.

B. Codeword Search and Selection

In this paper, the receiver chooses the precoding matrix (codeword) \mathbf{F} from a finite set of N possible codewords $\mathcal{F} = \{\mathbf{F}_1, \mathbf{F}_2, \dots, \mathbf{F}_N\}$ called the *codebook* shared by the transmitter and the receiver. The index of the selected codeword at the receiver, based on the knowledge of the channel, is fed back to the transmitter through a zero-delay limited capacity feedback channel. The codeword index $n = 1, 2, \dots, N$ is represented by b -bit binary ($N = 2^b$) resulting in b bits of feedback. We say that the codebook is *b-bit codebook* when it has $N = 2^b$ entries.

For beamforming, the optimal beamformer maximizes the effective SNR [4],

$$\mathbf{f} = \arg \max_{\mathbf{w} \in \mathcal{F}} \|\mathbf{H}\mathbf{w}\|_2^2. \quad (2)$$

The chordal distance between codeword vectors, \mathbf{f}_1 and \mathbf{f}_2 , is given by

$$d_{\text{ch}}(\mathbf{f}_1, \mathbf{f}_2) = \sin(\theta_{1,2}) = \sqrt{1 - |\mathbf{f}_1^* \mathbf{f}_2|^2}. \quad (3)$$

The chordal distance is used to analyze the distance property of the beamforming codebook. The beamformer selection can also be approximated by finding the beamforming vector from the codebook \mathcal{F} with the minimum chordal distance, to the right singular vector, \mathbf{v} , corresponding to the largest singular value of the channel \mathbf{H} [38].

For spatial multiplexing, the optimal unitary precoder is given by $\mathbf{F}_{\text{opt}} = [\mathbf{V}]_{1:M_s}$ where \mathbf{V} is the right singular matrix of the channel \mathbf{H} and $[\cdot]_{1:M_s}$ denotes the first to M_s -th columns of the given matrix [6], [38]. We employ the minimum singular value selection criteria (MSV-SC) [6]

$$\mathbf{F} = \arg \max_{\mathbf{W} \in \mathcal{F}} \lambda_{\min}\{\mathbf{H}\mathbf{W}\} \quad (4)$$

where λ_{\min} denotes the minimum singular value of the product $\mathbf{H}\mathbf{W}$. This selection criteria approximately maximizes the minimum substream SNR.

We use the two distance criteria proposed in [6] to evaluate the codebooks. It was shown in [6] that the codebook should be designed by maximizing either the minimum projection 2-norm distance, $\min_{\mathbf{F}_1 \neq \mathbf{F}_2} d_{p2}(\mathbf{F}_1, \mathbf{F}_2)$, or the minimum Fubini-study distance, $\min_{\mathbf{F}_1 \neq \mathbf{F}_2} d_{FS}(\mathbf{F}_1, \mathbf{F}_2)$. The projection 2-norm distance is defined as

$$\begin{aligned} d_{p2}(\mathbf{F}_1, \mathbf{F}_2) &= \|\mathbf{F}_1 \mathbf{F}_1^* - \mathbf{F}_2 \mathbf{F}_2^*\| \\ &= \sqrt{1 - \lambda_{\min}\{\mathbf{F}_1^* \mathbf{F}_2\}}, \end{aligned} \quad (5)$$

and the Fubini-study distance is defined as

$$d_{FS}(\mathbf{F}_1, \mathbf{F}_2) = \arccos |\det(\mathbf{F}_1^* \mathbf{F}_2)|. \quad (6)$$

In this paper, we will be concerned with selection criteria (2) for beamforming and (4) for spatial multiplexing, respectively. To analyze the distance properties of the codebook, (3) will be analyzed for beamforming and (5) and (6) are analyzed for spatial multiplexing.

III. KERDOCK CODEBOOK

In this section, we provide the background information on Kerdock codes and mutually unbiased bases (MUB), their construction, and their utility as a limited feedback codebook.

A. Preliminaries

Mutually unbiased bases (MUB) arises in connection with quantum information theory where the observable state of a quantum system is represented by the set of orthonormal bases (ONB) with certain correlation property (to be defined shortly) [39]. Some of the known MUB constructions are due to Alltop [40], Wootters and Fields [41], Klappenecker and Roetteler [42], Bandyopadhyay *et al.* [43], and recently by Gow [35]. Klappenecker and Roetteler [39] showed that many of the MUB constructions are equivalent and that these constructions have a close connection with complex projective space and uniform tight frames, both of which has been used for the construction and analysis of quantized codebooks for limited feedback MIMO systems. Based on these connections, we study the utility of MUB as a limited feedback codebook.

An MUB is a collection of two or more ONB with the property that the columns of different ONBs has the same correlation (or inner product). That is, if $\mathbf{S} = [\mathbf{s}_1 \dots \mathbf{s}_{M_t}]$ and $\mathbf{U} = [\mathbf{u}_1 \dots \mathbf{u}_{M_t}]$ are two $M_t \times M_t$ ONBs (*i.e.* $\mathbf{S}^* \mathbf{S} = \mathbf{I}_{M_t}$), the inner product of vectors drawn from each ONB satisfies the *mutually unbiased* property

$$|\langle \mathbf{s}_n, \mathbf{u}_m \rangle| = \frac{1}{\sqrt{M_t}} \quad (7)$$

for $n, m = 1, \dots, M_t$. MUB is the set $\mathcal{S} = \{\mathbf{S}_0, \mathbf{S}_1, \dots, \mathbf{S}_{N_s}\}$ with each ONB \mathbf{S}_n , $n = 0, \dots, N_s$, satisfying the mutually unbiased property. Two natural questions to ask are 1) how many such bases exists for a given dimension M_t , and 2) how to construct the MUB. First, the maximum number of ONBs, $N_s = |\mathcal{S}|$, has been shown to be $N_s \leq M_t + 1$ with equality if M_t is a power of a prime [42], [43]. It is presently unknown whether equality occurs when M_t is not a power of prime and this question remains to be an active area of research [35]. Second, several approaches for the construction of size $M_t + 1$ MUB for prime powers has been proposed (see [44] for an excellent survey and references therein). In this paper, we shall be concerned with power of two construction, mainly for $M_t = 2$ and 4, for its attractive construction and practical applicability. One drawback is that we do not have a construction for $M_t = 3$ with the finite alphabet properties so we shall differ this case for future work.

We now turn our attention to why MUB is a useful limited feedback codebook design. To see this, consider the beamforming codebook $\{\mathbf{f}_i\}_{i=1}^N \in \mathcal{F}$. The defining characteristics of MUB is its mutually unbiased property. That is, for $k, l = 1, \dots, N$, $|f_k^* f_l| = 0$ if \mathbf{f}_k and \mathbf{f}_l are chosen from the same bases and $|f_k^* f_l| = 1/\sqrt{M_t}$ if \mathbf{f}_k and \mathbf{f}_l are chosen from a different bases. We can compute the average inner product of the codebook as

$$\frac{1}{N(N-1)} \sum_{l=1}^N \sum_{l' \neq l}^N |\mathbf{f}_l^* \mathbf{f}_{l'}|^2 = \frac{N - M_t}{(N-1)M_t}. \quad (8)$$

The Grassmannian packing problem is to maximize the minimum pairwise distance of codewords using the distance function (3). Substituting (8) as the average inner product in (3), we obtain an approximate distance bound of the codebook

$$\begin{aligned} d_{\text{ch}}(\mathcal{F}) &\approx \sqrt{1 - \frac{N - M_t}{(N-1)M_t}} \\ &= \sqrt{\frac{N(M_t - 1)}{M_t(N-1)}} \end{aligned} \quad (9)$$

which is the same as the well known Rankin bound, an upper bound on the minimum distance for line packings [4]. Therefore, the MUB codebook is near optimal in the sense that the Rankin bound is met in an average sense.

B. Sylvester-Hadamard Codebook Construction

Kerdock codes originally proposed for error correcting codes [32] are known to be MUB [44]. MUB structure of Kerdock codes were used to design a scalable signature sequence for code division multiple access (CDMA) system by Heath *et al.* [33]. Kerdock codes are particularly attractive for implementation since the codebook contains finite alphabet, $\{\pm 1, \pm j\}$, and still satisfy the mutually unbiased property. We shall use the simplified Kerdock construction in [33] to construct $M_t = 2$ finite alphabet codebook. For $M_t = 4$, we first construct the Kerdock code and modify it to obtain the power construction described in Section III-C.

The Kerdock code construction proposed in [33] consists of $M_t = 2^B$ orthonormal matrices, where B is a positive integer. Orthonormal matrices are denoted \mathbf{S}_n , $n = 0, \dots, M_t - 1$, where each \mathbf{S}_n is a rotated Sylvester-Hadamard matrix. The key to the construction is the algebraic derivation of the rotating (or generator) matrices, \mathbf{D}_n , which does not rely on any search.

Let $\hat{\mathbf{H}}_{M_t}$ denote the size $M_t \times M_t$ Sylvester-Hadamard matrix where

$$\hat{\mathbf{H}}_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad (10)$$

and

$$\hat{\mathbf{H}}_{M_t} = \underbrace{\hat{\mathbf{H}}_2 \otimes \hat{\mathbf{H}}_2 \cdots}_{B \text{ times}}. \quad (11)$$

The general strategy for the Kerdock codebook construction is as follows:

- 1) Construct the diagonal matrices, \mathbf{D}_n , for $n = 0, 1, \dots, M_t - 1$. These are the generator matrices.
- 2) Each bases is constructed by $\mathbf{S}_n = (1/\sqrt{M_t})\mathbf{D}_n\hat{\mathbf{H}}_{M_t}$.
- 3) Let $\hat{\mathbf{F}} = [\mathbf{S}_0\mathbf{S}_1 \cdots \mathbf{S}_{M_t-1}]$.

For brevity, we omit the details of the diagonal generator matrix construction which can be found in [33].

C. Power Codebook Construction

Let p be a prime number and let $q = p^a$, where a is a positive integer. Let \mathbb{F} denote the finite field of order q^2 . We let G_q denote the finite group $\mathbb{F} \times \mathbb{F}$ of order q^4 with multiplication defined by

$$(a, b)(c, d) = (a + c, a^q c + b + d). \quad (12)$$

Lemma 1: Let α be an element of order $q + 1$ in \mathbb{F} . The mapping $\sigma : G_q \rightarrow G_q$ given by $\sigma(a, b) = (\alpha a, b)$ is an automorphism of order $q + 1$ of G_q .

Let χ be an irreducible character of G_q of degree q and X be an irreducible representation of G_q with character χ . The key result due to Gow [35] is the following.

Theorem 2: Let $p = 2$ and let X be of degree q consisting of unitary matrices. If D is a $q \times q$ matrix that satisfies $D^{q+1} = \mathbf{I}$ and $D^{-1}X(x)D = X(\sigma(x))$ for all x in G_q , then the powers $D, D^2, \dots, D^{q+1} = \mathbf{I}$ generates $q + 1$ pairwise mutually unbiased bases. Furthermore, all entries of D are in the field $\mathbb{Q}(\sqrt{-1})$.

Theorem 2 states that if a matrix D which satisfies the given conditions is found, then the powers of D generates the size $q + 1$ MUB.

For the limited feedback codebook design, Theorem 2 represents a powerful result when the number of transmit antennas are power of 2. Only the generating base D needs to be stored and the rest of the codebook can be generated from the products. Therefore

$$\mathcal{S} = \{S_0 = D, S_1 = D^2, \dots, S_{N_s-1} = D^{N_s}\}. \quad (13)$$

Note also the inclusion of identity element which corresponds to the case of antenna subset selection [34]. All of the previously proposed codebooks do not have identity as part of the codebook. In standards such as 3GPP LTE, the identity element is included in the codebook [8] for antenna selection. The MUB construction extends naturally for use in 3GPP LTE.

D. Codebook Arrangement

Once the MUB is generated, we apply the following procedure to arrange S_n into a codebook. For the beamforming system, we construct the composite matrix, $\hat{\mathbf{F}} = \begin{bmatrix} S_0 & S_1 & \dots & S_{N_s-1} \end{bmatrix}$ and define the codebook as the columns of $\hat{\mathbf{F}}$. That is

$$\mathcal{F} = \{\mathbf{F}_1 = [\hat{\mathbf{F}}]_1, \mathbf{F}_2 = [\hat{\mathbf{F}}]_2, \dots, \mathbf{F}_N = [\hat{\mathbf{F}}]_N\} \quad (14)$$

where $[\cdot]_n$ is used to denote the selection of n -th column of a given matrix and $N \leq M_t N_s$.

For the unitary precoding spatial multiplexing system, a subset of columns are selected from each \mathbf{S}_n to form the codebook. Notice that for a subset of columns drawn from a single \mathbf{S}_n , each columns are orthogonal to each other by construction. Hence, the design problem is to take a column subset from each \mathbf{S}_n so that the pairwise minimum distance, (5) or (6), is maximized and to see whether such codebooks can yield performance similar to previously proposed codebooks.

For M_s -stream spatial multiplexing codebook, take all M_s -column subsets from each \mathbf{S}_n . There are $\binom{M_t}{M_s}$ column subset combinations in each \mathbf{S}_n . The maximum number of codewords that the MUB can take is $N_s \times \binom{M_t}{M_s}$. We shall illustrate the system performance for a few possible subsets of so constructed codebook in Section VI. One way to select the subset is by identifying a unique column combinations from each \mathbf{S}_n so that every pairwise minimum distance (either (5) or (6)) is maximized. Unfortunately, exhaustive search appears to be the only way to find the best combination.

In Section V, we show that constant distance between codewords can be achieved by this construction, and in Section VI, we show through numerical simulation that this codebook performs comparable to same sized Grassmannian and Fourier based codebooks. We do not claim optimality of the so constructed spatial multiplexing codebook other than that it achieves full diversity.

E. Two Transmit Antenna Construction

In this section, we provide an example of Kerdock codebook construction for two antenna MIMO system. This is the trivial case for the Kerdock codebook construction and it is easy to verify that

$$\mathbf{D}_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \mathbf{D}_1 = \begin{bmatrix} 1 & 0 \\ 0 & j \end{bmatrix}. \quad (15)$$

The resulting \mathbf{S}_n are

$$\mathbf{S}_0 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \mathbf{S}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ j & -j \end{bmatrix}, \mathbf{S}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad (16)$$

where \mathbf{S}_0 is merely the scaled Sylvester-Hadamard matrix, $\hat{\mathbf{H}}_2$, with quaternary alphabet.

In this case we prefer to use the Sylvester-Hadamard construction since the power construction for $M_t = 2$ is

$$\mathbf{D} = \frac{1}{2} \begin{bmatrix} -1-j & -1+j \\ 1+j & -1+j \end{bmatrix} \quad (17)$$

is not exactly quaternary. Taking the powers of \mathbf{D} , we readily obtain the following MUB

$$\left\{ \frac{1}{2} \begin{bmatrix} -1-j & -1+j \\ 1+j & -1+j \end{bmatrix}, \frac{1}{2} \begin{bmatrix} -1+j & 1-j \\ -1-j & -1-j \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}. \quad (18)$$

This codebook does have the finite alphabet property and also includes the identity corresponding to antenna selection.

The beamforming codebook is constructed by forming the composite matrix $\hat{\mathbf{F}} = [\mathbf{S}_0 \ \mathbf{S}_1 \ \mathbf{S}_2]$ and considering each column as beamforming vectors,

$$\mathcal{F} = \{\mathbf{F}_1 = [\hat{\mathbf{F}}]_1, \mathbf{F}_2 = [\hat{\mathbf{F}}]_2, \dots, \mathbf{F}_N = [\hat{\mathbf{F}}]_6\}. \quad (19)$$

F. Four Transmit Antenna Construction

In this section, we give an example MUB codebook for $M_t = 4$. Our simulation results in Section VI utilize the codebook constructed in this example.

For $M_t = 4$, we start with the simplified Kerdock code construction in [33] and making a slight modification to one of the bases, we find that the following generator matrix satisfies Theorem 2

$$\mathbf{D} = \frac{1}{2} \begin{bmatrix} -j & -j & -j & -j \\ 1 & -1 & 1 & -1 \\ -j & -j & j & j \\ -1 & j & j & -j \end{bmatrix}. \quad (20)$$

By inspection, we can recognize that \mathbf{D} can be decomposed with a Sylvester-Hadamard matrix

as follows

$$\mathbf{D} = \frac{1}{2} \begin{bmatrix} -j & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -j & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} (\hat{\mathbf{H}}_2 \otimes \hat{\mathbf{H}}_2). \quad (21)$$

Finally, computing $\mathbf{S}_n = \mathbf{D}^{n+1}$ yields

$$\begin{aligned} \mathbf{S}_0 &= \frac{1}{2} \begin{bmatrix} -j & -j & -j & -j \\ 1 & -1 & 1 & -1 \\ -j & -j & j & j \\ -1 & 1 & 1 & -1 \end{bmatrix} \\ \mathbf{S}_1 &= \frac{1}{2} \begin{bmatrix} -1 & -1 & -j & j \\ -j & -j & -1 & 1 \\ -j & j & -1 & -1 \\ 1 & -1 & j & j \end{bmatrix} \\ \mathbf{S}_2 &= \frac{1}{2} \begin{bmatrix} -1 & j & j & 1 \\ -1 & j & -j & -1 \\ j & -1 & -1 & -j \\ -j & 1 & -1 & -j \end{bmatrix} \\ \mathbf{S}_3 &= \frac{1}{2} \begin{bmatrix} j & 1 & j & -1 \\ j & -1 & j & 1 \\ j & 1 & -j & 1 \\ j & -1 & -j & -1 \end{bmatrix} \\ \mathbf{S}_4 &= \mathbf{I}_4. \end{aligned} \quad (22)$$

Thus we obtain an MUB with quaternary alphabet based on Kerdock codes.

For the beamforming system, we construct the composite matrix from the above construction,

$$\hat{\mathbf{F}} = \begin{bmatrix} \mathbf{S}_0 & \mathbf{S}_1 & \mathbf{S}_2 & \mathbf{S}_3 & \mathbf{S}_4 \end{bmatrix} \text{ and define the codebook as the columns of } \hat{\mathbf{F}}. \text{ That is}$$

$$\mathcal{F} = \{\mathbf{F}_1 = [\hat{\mathbf{F}}]_1, \mathbf{F}_2 = [\hat{\mathbf{F}}]_2, \dots, \mathbf{F}_N = [\hat{\mathbf{F}}]_{20}\}, \quad (23)$$

for a $N = 20$, 5-bit codebook. The identity element, \mathbf{S}_4 , can be deleted if antenna selection is not needed.

For the unitary precoding spatial multiplexing system, we shall take all M_s column subset from each \mathbf{S}_n , $n = 0, 1, \dots, 4$. For $M_s = 2$, we obtain $5 \times \binom{4}{2} = 30$ codewords, or 5-bit codebook, and for $M_s = 3$, $5 \times \binom{4}{3} = 20$ codewords, or 5-bit codebook. The distance properties of the so obtained codebooks are analyzed in Section V.

We have thus obtained a finite alphabet codebook which can be shared for beamforming and spatial multiplexing. Next, we will analyze and quantify the storage and search complexity associated with Kerdock codes and compare it with Grassmannian and Fourier based construction.

IV. CODEBOOK STORAGE AND SEARCH COMPLEXITY

Previous limited feedback codebook designs were primarily concerned with achievable performance [4], [6], but the resulting codebook was such that the codebooks of various sizes for different modes of transmission had to be stored at both the transmitter and the receiver. In today's hand-held mobile devices, power consumption and device size are two major design challenges [45], [46]. A large portion of today's baseband devices are memory elements while power consumption can be related to the amount of computation required on the device. To minimize the impact of codebook based limited feedback implementation, it is of great interest to minimize the codebook storage and search computation for various modes of transmission. Furthermore, in recent standards such as 3GPP LTE and 3GPP2 UMB, up to 300km/h of mobility is being considered [8]. A simple search algorithm is of great interest to reduce the time of adaptation in highly mobile environment. In this section, we quantify the storage and search complexity of the proposed MUB codebook and compare it with Grassmannian and Fourier based designs.

A. Storage

To estimate the storage requirements, we will consider the number of real elements (*i.e.* two real components for one complex value) to store a codebook for each mode of transmission (*i.e.* beamforming and spatial multiplexing). Let N_b denote the number of bits available in the system to represent a real number. It is easy to see that the storage required for a single codebook with N entries of $M_t \times M_s$ complex-entry codewords for a particular transmission mode is upper bounded by $2N_bNM_tM_s$ -bits. Note that some reduction in number of stored bits may be possible due to specific values taken on by the codeword entries, but we will only contend with the worse case scenario.

The Grassmannian codebook [4], [6], [47] does not yield any systematic construction so the entire codebook, element by element, must be stored. Thus, the storage requirement is $2N_bNM_tM_s$ -bits for each codebook.

The Fourier based codebook [27] requires the storage of one diagonal generator matrix (*i.e.* M_t complex entries) and $M_t \times M_s$ entries of a DFT matrix. In general, the generator matrix is different for each mode of transmission but the $M_t \times M_t$ DFT matrix can be used for all cases. The storage requirement for Fourier based codebook is $2N_b(M_t + M_tM_s)$ -bits for a given transmission mode. Note that the storage requirement is independent of the codebook size, N , because the generator matrix is designed for a particular codebook size N .

The MUB codebook construction in Section III requires storage of only the generating bases \mathbf{D} . For $M_t = 2$, with Sylvester-Hadamard construction, we only need to store the Hadamard matrix, 4 bits, and \mathbf{D}_1 which has two entries from 2-bit quaternary alphabet, for a total of 8 bits. With the power construction for $M_t = 2$, each entry of \mathbf{D} can be stored with 2-bits, indicating the sign of the real and imaginary parts. The total storage required is again 8 bits. For $M_t = 4$ construction, with the Hadamard decomposition as in (21), we need 4×2 bits for the diagonal matrix (where each quaternary alphabet is represented by 2 bits) and 4 bits for the Hadamard matrix (since the entries are reals only). Therefore, the total required storage is 12 bits. Note that the MUB codebook storage is independent of N_b and the same codebook can be used for beamforming and spatial multiplexing.

For a fair comparison, Table II shows the number of bits required to store the Kerdock, Fourier, and Grassmannian codebook for $M_t = 4$ using $N = 16$ for beamforming and $N = 8$

for 2-stream unitary precoded spatial multiplexing. The Kerdock codebook results in significant storage savings.

B. Search Complexity

For search complexity, we will consider the number of arithmetic computation required to arrive at the desired codeword. We assume that (2) is tested for beamforming and (4) is tested for spatial multiplexing with the estimated channel matrix. Since the norm computations are common for all codebook entries, we compare the computation required to compute $\mathbf{H}\mathbf{f}$ for (2) and $\mathbf{H}\mathbf{F}$ for (4) for each codeword in the codebook. Our search strategy is exhaustive in which the effective channel gain is computed for all the codewords in the codebook and the codeword with the largest effective channel gain is selected as the suboptimal choice. Search space reduction using some of the well known methods in VQ literature may be possible [10] but we will defer this for future work. In this paper, we will be concerned with reduction in the arithmetic computation due to finite alphabet construction of MUB codebooks.

For the sake of illustration, consider the computation of $\mathbf{H}\mathbf{f}$ for the 2×2 beamforming case. We want to compute

$$\begin{aligned} \mathbf{H}\mathbf{f} &= \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \\ &= \begin{bmatrix} h_{11}f_1 + h_{12}f_2 \\ h_{21}f_1 + h_{22}f_2 \end{bmatrix}. \end{aligned} \quad (24)$$

For previously known codebooks, the entries of \mathbf{f} are complex valued thus requiring 4 complex multiplies and 2 complex additions. The Kerdock codebook entries are $\{\pm 1, \pm j\}$ which reduces the complex multiplication into either a sign change when $f_i = \pm 1$, or swapping the real and imaginary part with sign change when $f_i = \pm j$. Sign change is a trivial operation in digital systems and swapping the real and imaginary part can be accomplished by reading opposite entries in the memory. Therefore, the Kerdock code effectively achieves multiplier-less computation of $\mathbf{H}\mathbf{f}$ and $\mathbf{H}\mathbf{F}$.

Table III shows the required number of arithmetic computation at the receiver for codeword selection. For beamforming, the Grassmannian and Fourier based codebooks require NM_tM_r complex multiplies and $NM_r(M_t - 1)$ complex additions to find all the candidate effective

SNRs. Meanwhile, the proposed Kerdock codebook does not require any complex multiplication as described above. Similar elimination of complex multiplication is obtained for spatial multiplexing. Depending on the specific implementation for baseband processing, this could translate to reduced cycle times for complex arithmetic logic units (ALU) or a possibility for a hard wired logic implementation. Thus, for a mobile terminals with limited computational resource and memory, the Kerdock codebook is an attractive solution for implementation.

V. RELATIONSHIP WITH PREVIOUS DESIGNS

In this section, we provide the distance, diversity, and capacity analysis of the MUB codebook with comparisons to Grassmannian and Fourier based codebooks.

A. Distance Properties

The distance properties are one of the characteristics of MUB design. The following lemma captures the fact that pairwise chordal distance between any codewords in the MUB codebook can only take two values.

Lemma 3: For any pair of beamforming MUB codebook elements \mathbf{f}_k and \mathbf{f}_l , $k, l = 1, 2, \dots, N$, in (14), the chordal distance (3) is

$$d_{\text{ch}}(\mathbf{f}_k, \mathbf{f}_l) = \begin{cases} 1 \\ \sqrt{1 - \frac{1}{M_t}} \end{cases} \quad (25)$$

Proof: If \mathbf{f}_k and \mathbf{f}_l are from the same ONB, $|\mathbf{f}_k^* \mathbf{f}_l|^2 = 0$ since the columns are orthonormal by construction. If \mathbf{f}_k and \mathbf{f}_l are from a different ONB, $|\mathbf{f}_k^* \mathbf{f}_l|^2 = 1/M_t$ by the mutually unbiased property. ■

It is interesting to observe that as $M_t \rightarrow \infty$, the codebook approaches that of the Grassmannian codebook.

For spatial multiplexing, we show that the beamforming Kerdock codebook can be arranged so that the spatial multiplexing codebook with a large pairwise chordal distance can be derived.

Let us consider the proposed spatial multiplexing codebook derived from the beamforming codebook and examine the projection 2-norm and Fubini-study distances. First, consider the $M_s = 2$ spatial multiplexing codebook for $M_t = 4$.

Property 4: Let \mathbf{F}_k and \mathbf{F}_l , $k \neq l$, be 4×2 matrices composed by taking two columns from any power of \mathbf{D} . Then,

$$|\det(\mathbf{F}_k^* \mathbf{F}_l)| = \begin{cases} 0 \\ 1/\sqrt{M_t} \end{cases} \quad (26)$$

Proof: Each \mathbf{F}_k and \mathbf{F}_l can be written as $\mathbf{F}_k = \mathbf{D}^p \mathbf{E}_k$ and $\mathbf{F}_l = \mathbf{D}^q \mathbf{E}_l$ where \mathbf{E}_k and \mathbf{E}_l are 4×2 column selection matrices. Then

$$\begin{aligned} \mathbf{F}_k^* \mathbf{F}_l &= \mathbf{E}_k^T \mathbf{D}^{q*} \mathbf{D}^p \mathbf{E}_l \\ &= \mathbf{E}_k^T \mathbf{D}^r \mathbf{E}_l, \end{aligned} \quad (27)$$

where $r = (q - p)^*$ when $q > p$ and $r = (p - q)$ when $p > q$. Due to the construction \mathbf{D}^r is one of the member bases. The act of \mathbf{E}_k^T and \mathbf{E}_l takes 2×2 submatrix of \mathbf{D}^r . Any member \mathbf{D}^r has a structure such that any 2×2 submatrix selected this way always contains 1) all reals, 2) a pair of reals and a pair of imaginary, or 3) all imaginary, from the quaternary alphabet. It is easy to verify, by listing all possibilities, that the determinant of such 2×2 matrix can only take values 0 or $1/\sqrt{M_t}$. ■

Now consider selecting three columns from each \mathbf{S}_n to construct a $M_s = 3$ spatial multiplexing codebook.

Property 5: Let \mathbf{F}_k and \mathbf{F}_l , $k \neq l$, be 4×3 matrices by selecting any 3 columns from each \mathbf{S}_n . Then,

$$|\det(\mathbf{F}_k^* \mathbf{F}_l)| = 1/\sqrt{M_t}. \quad (28)$$

Proof: The $\det(\mathbf{F}_k^* \mathbf{F}_l)$ is given by the determinant of a 3×3 submatrix of some bases \mathbf{S}_n . Recall that the adjoint of a square matrix \mathbf{D} , denoted $\text{adj}(\mathbf{D})$, is given by $\text{adj}(\mathbf{D}) = \mathbf{D}^{-1} \cdot \det(\mathbf{D})$. Since \mathbf{D} is unitary, $\mathbf{D}^{-1} = \mathbf{D}^*$ and $\det(\mathbf{D}) = \mathbf{I}$. So, $\text{adj}(\mathbf{D}) = \mathbf{D}^*$. Therefore, the adjoint matrix also has quaternary alphabet. The elements of adjoint matrix is the cofactors which are minors, or determinant of 3×3 submatrix, with appropriate signs. This shows that every determinant of 3×3 submatrix is in the set $\{\pm 1, \pm j\}$ with scaling $1/\sqrt{M_t}$ and the result follows. ■

Since the projection 2-norm increases as the minimum singular value of $\mathbf{F}_1^* \mathbf{F}_2$ is decreased, we see that the projection 2-norm is maximized when we have the column selection such that $|\det(\mathbf{F}_1^* \mathbf{F}_2)| = 0$, or the product matrix is singular. The Fubini-Study distance (6) also maximized by minimizing $|\det(\mathbf{F}_1^* \mathbf{F}_2)|$. A 3-bit Kerdock codebook for $M_s = 2$ spatial multiplexing in Table

It was found by inspecting the column selections which resulted in the largest projection 2-norm distance. Note that such subset selection is not possible with other codebook constructions because the subset matrix cannot be guaranteed to be a unitary matrix.

B. Diversity

The diversity order is an important performance metric which indicates the probability of symbol error trends for high SNR regions. The Kerdock codebook arranged as in (14) is easily verified to have full rank.

Theorem 6: If $N \geq M_t$, the Kerdock codebook has full diversity order.

Proof: The proof follows that found in [4] using the fact that the Kerdock codebook is of full rank since it is composed of unitary matrices. Thus, maximum diversity is achieved by the Kerdock codebook. ■

C. Capacity

The system capacity associated with quantized codebook is an important indicator of the quality of the codebook [4], [12]. The capacity of the system with a precoder is usually written as

$$C(\mathbf{F}) = E_{\mathbf{H}} \left[\log_2 \det \left(\mathbf{I}_{M_s} + \frac{\mathcal{E}_s}{M_s N_o} \mathbf{F}^* \mathbf{H}^* \mathbf{H} \mathbf{F} \right) \right]. \quad (29)$$

where $E_{\mathbf{H}}$ denotes the expectation with respect to \mathbf{H} . This is the achievable upper bound when there are no channel estimation errors and feedback delay, but not the true capacity since power allocation (*i.e.* water filling solution) is not considered. To give a fair comparison, achievable capacity of an equal size Grassmannian, Fourier, and Kerdock codebook are compared in Fig. 2. For the beamforming case (dashed line), we can see that the Grassmannian, Fourier based, and Kerdock codes have the same achievable capacity with approximately 1.5dB of loss compared to perfect CSIT. The spatial multiplexing case (solid line) shows all 5-bit codebooks provide nearly identical capacity. The 3-bit Grassmannian codebook capacity for spatial multiplexing system is also shown to illustrate the loss due to codebook size reduction to 3-bits. Therefore, there is essentially no capacity loss in using the Kerdock codebook.

VI. SIMULATION RESULTS

In this section, we give numerical simulation results comparing 1) Vector Symbol Error Rate (VSER) performance of limited feedback beamforming system, and 2) VSER performance of two stream unitary precoded spatial multiplexing system for the Grassmannian, Fourier, and Kerdock codebook. All simulations are performed for $M_t = M_r = 4$ assuming delay-free feedback. No forward error correction is used.

Experiment 1

The limited feedback beamforming system VSER performance is shown in Fig. 3. For modulation, 64-QAM is used and maximum ratio combining (MRC) is used at the receiver. The ideal beamforming result is the lower bound when perfect CSIT is available. The Kerdock codebook outperforms both Grassmannian and Fourier based codebooks.

Experiment 2

The VSER performance for limited feedback two-stream unitary precoded spatial multiplexing system using 5-bit codebook is shown in Fig. 4. A 16-QAM modulation and zero-forcing receiver is used. The Kerdock codebook outperforms the Grassmannian codebook and the Fourier based codebook, despite having only 30 entries.

To clearly see the performance difference among the codebook designs, Fig. 5 shows the SNR gap between the ideal CSIT case and the three codebook designs at $VSER = 10^{-2}$. As expected, the Grassmannian codebook outperforms the Fourier codebook. The Kerdock codebook shows worse performance for 3-bit codebook because only 8 of 30 possible codewords are used. However, as we increase the codebook size to 4 and 5 bits, the Kerdock codebook outperforms the Grassmannian codebook which is quite remarkable considering the fact that the codebook contains only quaternary alphabet.

Overall, the results indicate that the Kerdock codebook can perform comparable or better than previously known codebooks with additional benefit of 1) structured construction, 2) finite alphabet, 3) reduced search complexity, and 4) shared codebook between beamforming and spatial multiplexing.

VII. CONCLUSION

In this paper, we proposed to use Kerdock codes for limited feedback precoded MIMO systems. The Kerdock code is an MUB set with correlation properties that can be linked to

Grassmannian line packing problem, equiangular frames, and Welch bound equality sequence sets. We showed that the Kerdock codes can achieve full diversity and performance comparable or better than previously known codebooks with additional benefits of finite alphabet construction, reduced storage and search requirements, and shared codebook between beamforming and spatial multiplexing. We found that there is essentially no loss in achievable capacity compared to equal size Grassmannian and Fourier based codebooks. One limitation of this work is that the Kerdock codes can only be constructed for number of transmit antennas which are power of two. An open problem remains in constructing odd dimension codebook with finite alphabet entries. Our future work will consider effects of space or time correlated channels and possible extensions to multiuser scenarios. In particular, Kerdock codes are possibly applicable to a multiuser MIMO system using a unitary basis sets, known as PU^2RC [48], [49].

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TABLE I: Unitary precoding codebook for 2 streams derived from the beamforming codebook. The column selection in the subscript is chosen to maximize the pairwise projection 2-norm distance.

$[\mathbf{S}_0]_{1,2}$	$[\mathbf{S}_0]_{3,4}$	$[\mathbf{S}_1]_{1,3}$	$[\mathbf{S}_1]_{2,4}$
$[\mathbf{S}_2]_{1,4}$	$[\mathbf{S}_2]_{2,3}$	$[\mathbf{S}_3]_{1,4}$	$[\mathbf{S}_3]_{2,3}$

TABLE II: Number of bits required for storing MUB, Fourier, and Grassmannian codebooks for $M_t = 4$ and using $N = 16$ for beamforming and $N = 8$ for 2-stream spatial multiplexing. A system dependent number of bits used to represent a real number is denoted by N_b .

MUB	Fourier	Grassmannian
12	$40N_b$	$256N_b$

TABLE III: Comparison of computational requirement for codeword selection

Beamforming Selection		
	Grassmannian or Fourier	Kerdock
Multiply	$N M_t M_r$	0
Addition	$N M_r (M_t - 1)$	$N M_r (M_t - 1)$
Spatial Multiplexing: Projection 2-norm		
	Grassmannian or Fourier	Kerdock
Multiply	$N M_s M_r^2$	0
Addition	$N M_r^2 (M_s - 1)$	$N M_r^2 (M_s - 1)$
Spatial Multiplexing: Fubini-Study		
	Grassmannian or Fourier	Kerdock
Multiply	$N M_s^2 M_r$	0
Addition	$N M_s^2 (M_r - 1)$	$N M_s^2 (M_r - 1)$

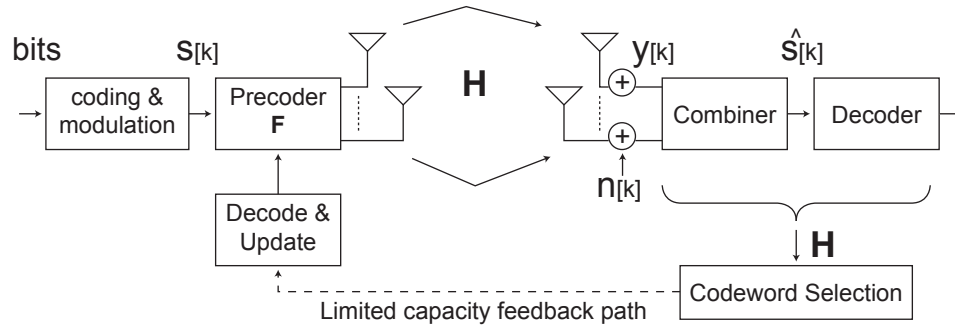
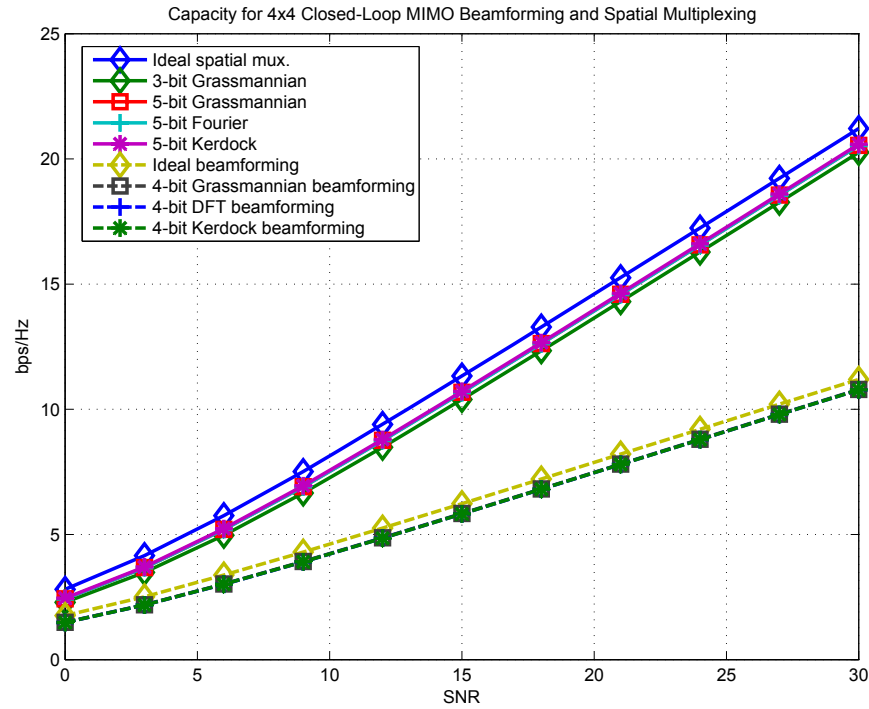


Fig. 1: Block diagram of general limited feedback MIMO System

Fig. 2: Capacity of $M_t = M_r = 4$ beamforming system and unitary precoded spatial multiplexing system using perfect CSI and the Kerdock codebook

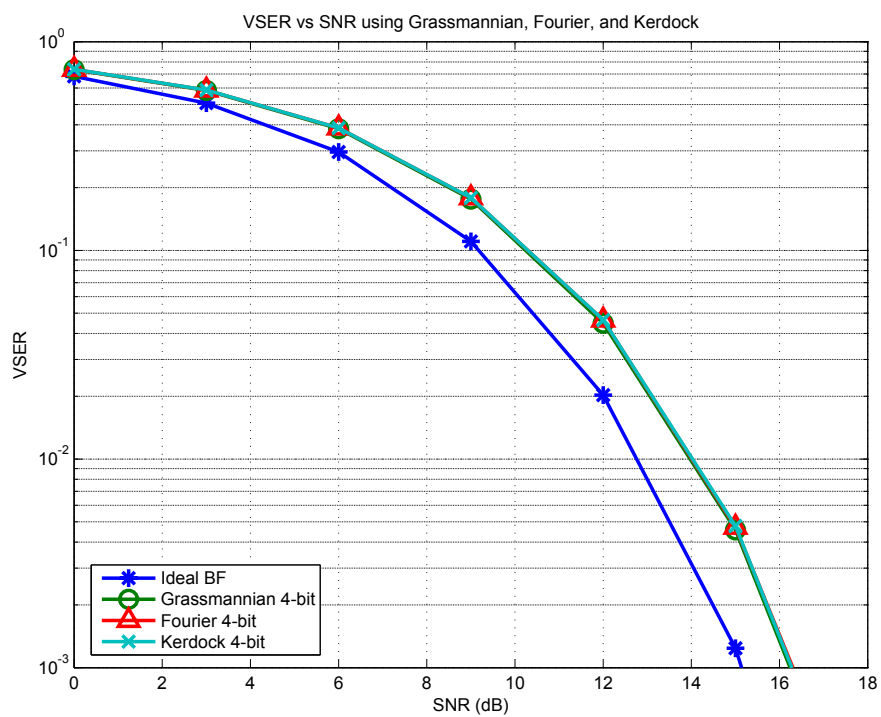


Fig. 3: Vector Symbol Error Rate performance of $M_t = M_r = 4$ beamforming system using 64-QAM comparing perfect CSI, Grassmannian codebook, and Kerdock codebook

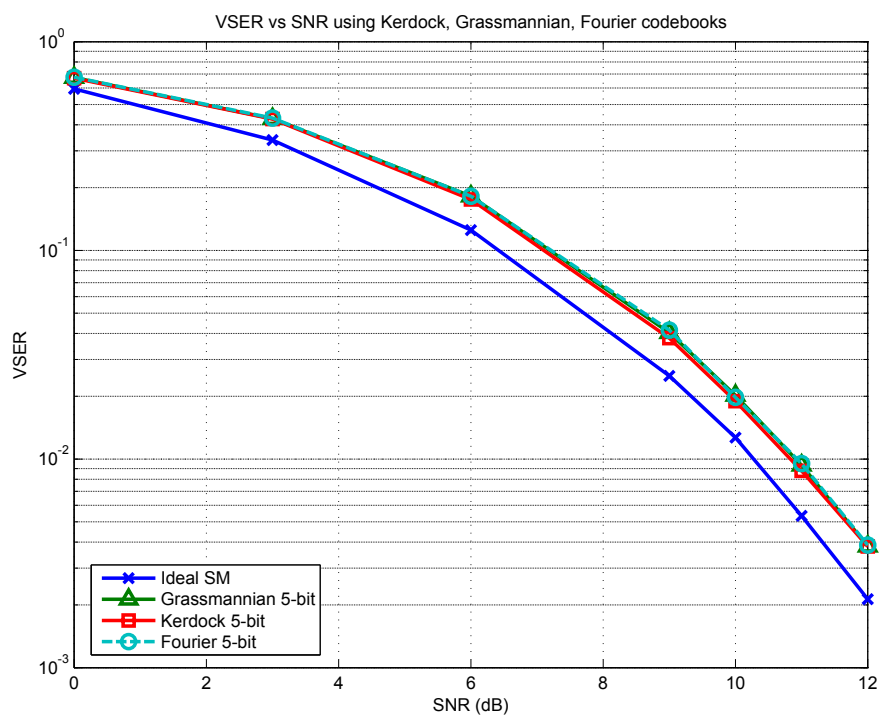


Fig. 4: Vector Symbol Error Rate Performance of $M_t = M_r = 4$ Spatial Multiplexing System 16-QAM comparing perfect CSI, Grassmannian codebook, and Kerdock codebook

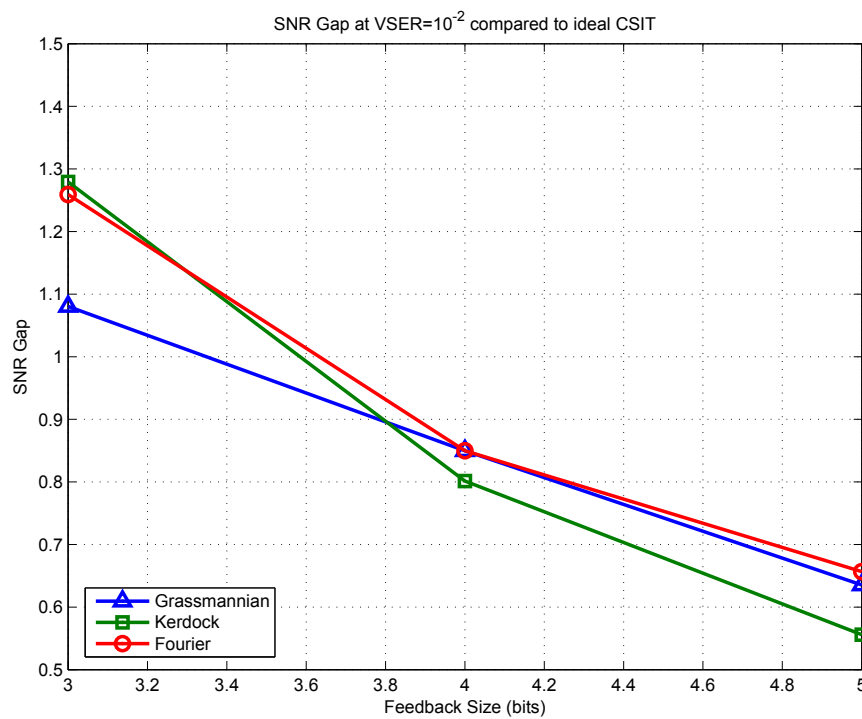


Fig. 5: SNR gap between ideal CSI case and various codebook design for 2-stream Spatial Multiplexing System at VSER = 10^{-2}